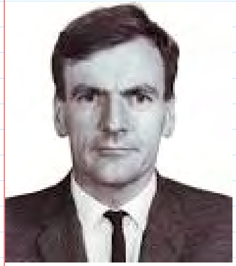


Renfrey Potts(1925-2005)

Return to Ising:

$G=(V,E)$ - finite graph

$\lambda: V \to \{-1,1\}$ - spin configuration.

$$P_\beta(\gamma) = \frac{1}{Z_\beta} e^{-2\beta \sum_{x\sim y} 1_{\{\lambda(x)\neq\lambda(y)\}}}$$

Generalization: q-state Potts:

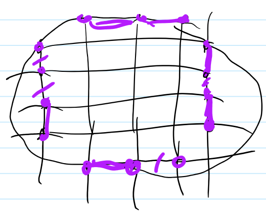
$\lambda: V \to \{1,2,\dots,q\}$

$$P_{\beta,q}(\lambda) = \frac{1}{Z_{\beta,q}} e^{-2\beta \sum_{x\sim y} 1_{\{\lambda(x)\neq\lambda(y)\}}}.$$

Natural graph: $\mathbb{Z}^2$ (finite subset of)

Works for other graphs too.

Add boundary conditions: $\overset{\text{boundary}}{B} \subset V$, only consider with given values on B.



Another representation: Fortuin - Kasteleyn model,

random clusters.

Cees Fortuin

Pieter Kasteleyn(1924-1996)

Configuration:

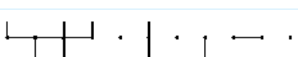
$\eta: E \to \{0,1\}$

Interpretation:

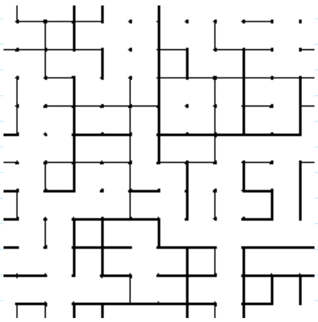
$$\eta(e) = \begin{cases} 0 - \text{closed} \\ 1 - \text{open} \end{cases}$$

$$P_{p,q}(\eta) = \frac{1}{Z_{p,q}} p^{\sum \eta(e)} (1-p)^{\sum(1-\eta(e))} q^{\# k(\eta)}$$

$k(\eta)$ - number of connected clusters of open edges



$\kappa(\eta)$ — number of connected clusters of open edges



$q=1$ — again, percolation.

Boundary condition:

fix condition of all edges between vertices of $\mathcal{B}$.

$\tilde{\kappa}(\eta)$ — number of clusters not intersecting $\mathcal{B}$.

$$P_{p,q}(\eta) = \frac{1}{Z_{p,q}} p^{\sum \eta(e)} (1-p)^{\sum(1-\eta(e))} q^{\tilde{\kappa}(\eta)}$$

all nonfixed edges.

Couple two models:

Define γ and η on same probability space:

$\lambda: V\backslash\mathcal{B} \to \{0, \ldots q\}$

$\eta: E\backslash E^b \to \{0,1\}$.

$\varphi := (\lambda, \eta)$.

$$P_{p,q}(\varphi) = \frac{1}{Z_{p,q}} p^{\sum_e \eta(e)} (1-p)^{\sum_e (1-\eta(e))} \prod_e \mathbb{1}_{\{(\lambda(x)-\lambda(y))\eta(e)=0\}}, \quad e=(x,y)$$

In other words, select the value on each vertex at random, then only assign λ which can only join vertices with the same value.

Robert G. Edwards(1925-2013)

Alan Sokal

Theorem (Edwards - Sokal)

Let $p = 1 - e^{-2\beta}$

Then for each λ_0:

$$P(\lambda_0) := \sum_{\substack{\varphi = (\lambda, \eta) \\ \lambda = \lambda_0}} P_{p,q}(\varphi) = \overbrace{\frac{1}{Z_{\beta,q}} e^{-2\beta \sum_{x \sim y} \mathbb{1}\{\lambda_0(x) \neq \lambda_0(y)\}}}^{q\text{-Potts probability}}$$

For each η_0

$$P(\eta_0) = \sum_{\substack{\varphi = (\lambda, \eta) \\ \eta = \eta_0}} P_{p,q}(\varphi) = \frac{1}{Z_{p,q}} p^{\sum \eta(e)} (1-p)^{\sum (1 - \eta(e))} q^{\tilde{k}(\eta)}$$

In other words, if we choose η according to FK, then assign values $\{1, \dots, q\}$ to each cluster uniformly at random, we get Potts.

Proof. For edges: for fixed η_0, need to sum up the probabilities of all compatible λ.

Each connected cluster should have the same value assigned, so we have $q^{\tilde{k}(\eta)}$ choices.

(Clusters intersecting the boundary are already assigned).

For vertices: for fixed λ_0, compatible η

if $e = (x,y)$, $\lambda_0(x) \neq \lambda_0(y)$, then $\eta(e) = 0$.

All other edges can be both open or closed

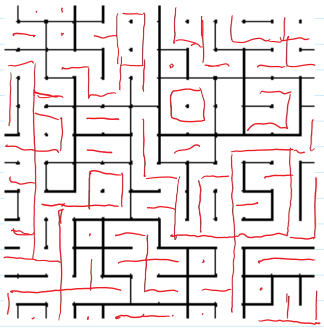
So the relative weight is

$$(1-p)^{\#\{e : \lambda(x) \neq \lambda(y)\}} \prod_{\substack{e = (x,y) \\ \lambda(x) = \lambda(y)}} (p + (1-p)) =$$

$$(1-p)^{\#\{e : \lambda(x) \neq \lambda(y)\}} = e^{-2\beta \#\{e : \lambda(x) \neq \lambda(y)\}}$$

Duality:

Each random cluster configuration λ also defines dual configuration λ^* on the dual lattice a dual edge is open iff an edge is closed.



Lemma. λ^* has law of p^*, q - FK-model, where $\frac{p^*}{1-p^*} = \frac{1-p}{p} q$.

Proof. Euler formula:

$\#\text{open} + \#\text{clusters} - \#\text{dual clusters} = \text{const}.$

Let $\sim$ means "proportional with a coefficient independent of configuration"

Then $p^{\#\text{open}} (1-p)^{\#\text{closed}} q^{\#\text{clusters}} \underset{\substack{\text{divide by} \\ p^{\#E}}}{\sim} \left(\frac{1-p}{p}\right)^{\#\text{closed}} q^{\#\text{clusters}} \underset{\text{Euler}}{\sim}$
$\left(\frac{1-p}{p} q\right)^{\#\text{closed}} q^{\#\text{dual clusters}} = \left(\frac{p^*}{1-p^*}\right)^{\#\text{dual open}} q^{\#\text{dual clusters}} \sim$
$p^{*\,\#\text{dual open}} (1-p^*)^{\#\text{dual closed}} q^{\#\text{dual clusters}}$

Special value: $p = p^*$.

$p_{sd}(q) = \frac{\sqrt{q}}{\sqrt{q}+1}$ $\qquad q=1 \quad \boxed{p_{sd} = \frac{1}{2}.}$

self dual $\qquad$ Kesten.

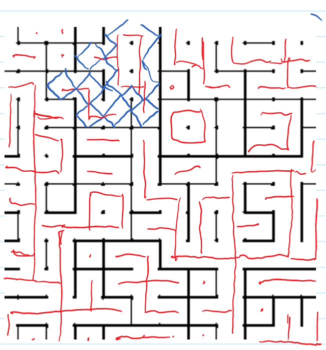
Vincent Beffara

Theorem (Beffara - Duminil-Copin).

p_{sd} is critical on $\mathbb{Z}^2$.

(i.e. for $p < p_{sd}$ - a.s. no infinite component, $p > p_{sd}$ - a.s. infinite component)

Double Loop representation.



On medial lattice: collection of loops separating dual and direct clusters.

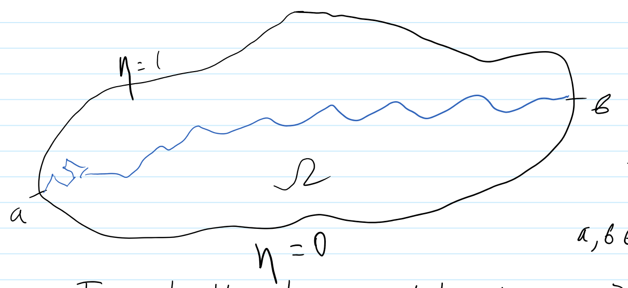
Weight of configuration:

$\left(\frac{p}{1-p} \frac{1}{\sqrt{q}}\right)^{\#\text{open}} (\sqrt{q})^{\#\text{loops}}.$

For $p = p_{sd}$: $(\sqrt{q})^{\#\text{loops}}$

[illegible]

Dobrushin boundary conditions:



Ω - simply connected, $a, b \in \partial\Omega$

In double loop model, there is a path γ from a to b - not a loop.

Conjecture. For $q \in [0,4]$, the law of γ converges to SLE_κ, $\kappa = \frac{4\pi}{\arccos(-\frac{\sqrt{q}}{2})}$

Thm (Smirnov) Correct for $q=2$ (Ising), $\left(\kappa = \frac{16}{3}\right)$.

$3 \cdot \frac{16}{3} = 16$

∂SLE_κ "=" SLE

Duplantier duality

$\kappa > 4$.

D. Zhang

$^{16}/_{\kappa}$

Dubedat.

Observable (for all q).

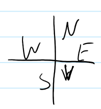
Defined on medial lattice to medial lattice!

$\sigma := 1 - \frac{2}{\pi} \arccos\left(\frac{\sqrt{q}}{2}\right)$.

$$F(z) := \frac{1}{Z_q} \sum_{z \in \gamma} (\sqrt{q})^{\# \text{loops}} e^{i\sigma W_{\gamma_{z \to b}}} = E\left(e^{i\sigma W_{\gamma_{z\to b}}}\right)$$

We consider all configurations with $\gamma \ni z$

Key lemma (for double-loop model).



Let v be any vertex, N, E, S, W - four adjacent edges. Then

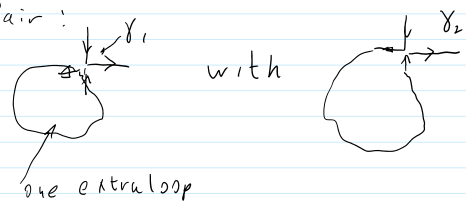
$F(N) - F(S) = i(F(E) - F(W))$

Discrete Cauchy-Riemann: compare with $\frac{\partial f}{\partial x} = i \frac{\partial f}{\partial y}$.

Proof.

Let γ enters v on N, exits on E.

Pair:



Let W_0 - weight of γ_2 (i.e. probability that γ_2 is the interface). Then weight of γ_1 is $\sqrt{q}\, W_0$ - one extra loop.

Then contributions of γ_2 and γ_1 to each function are

	$F(N)$	$F(E)$	$F(S)$	$F(W)$
γ_1	$\sqrt{q}\, W_0$	$e^{-i\frac{\pi}{2}\sigma}\sqrt{q}\, W_0$	0	0
γ_2	W_0	$e^{-i\frac{\pi}{2}\sigma} W_0$	$e^{-i\pi\sigma} W_0$	$e^{i\frac{\pi}{2}\sigma} W_0$

Some it up:

$$F(N) - F(S) = \sqrt{q}\, W_0 + W_0 - e^{-i\pi\sigma} W_0 = \left(\sqrt{q} + 1 - e^{-i\pi\sigma}\right) W_0$$

$$F(E) - F(W) = \left((\sqrt{q}+1) e^{-i\frac{\pi}{2}\sigma} - e^{i\frac{\pi}{2}\sigma}\right) W_0$$

Easy computation: $F(N) - F(S) = i\left(F(E) - F(W)\right)$

Again: not enough to determine F: each edge in two equations, each equation involves four edges.

Conjecture: Let Ω be a simply connected domain, $a, b \in \partial\Omega$, $\varphi : (\Omega, a, b) \to (\mathbb{R} \times (0,1), -\infty, \infty)$.
Let $G^\delta = (V^\delta, E^\delta)$ be $\delta \mathbb{Z}^2$ approximation of Ω, a_δ, b_δ - approximation of a and b.
Then, for some $C(q)$, $C(q)\delta^{-\sigma} F_q^\delta(z) \to \varphi'(z)^\sigma$ when $\delta \to 0$.

Theorem (Smirnov). Holds for Ising: $q = 2$, $\sigma = \frac{1}{2}$.

We'll prove later in the course.